Analytical bounds for non-asymptotic asymmetric state discrimination

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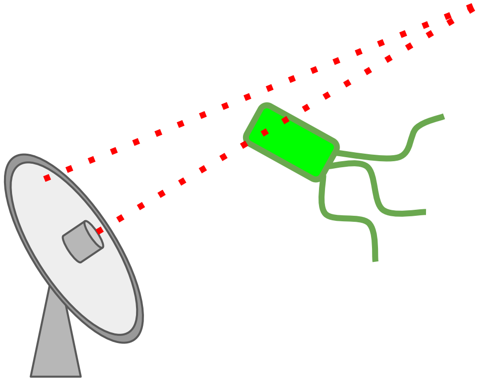
Why is quantum state discrimination important?

- State discrimination = choosing which quantum state we have, from some finite set.
- Many physical experiments can be regarded as state discrimination.
- Processes can be modelled as quantum operations and probes can be described as quantum states.
- We want to discriminate between possible outputs.
- Example: quantum target detection.
- Example: probing a substance with photons to find transmission.



Why does state discrimination have ultimate bounds?

- Better measurement devices perform better measurements!
- Classically, we can always perform an arbitrarily good measurement.
- We cannot perfectly distinguish between non-orthogonal quantum states.
- There is an ultimate bound on quantum state discrimination.

$$\rho_1 = |0\rangle\,, \quad \rho_2 = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Symmetric vs asymmetric state discrimination

- Focus on binary discrimination.
- Two types of errors can occur: false-positives and false-negatives.
- Symmetric discrimination = minimise the average error probability.
- Limited by the Helstrom bound.

$$T = \|\rho_1 - \rho_2\|_1, \quad p_E = \frac{1}{2}\left(1 - \frac{1}{2}T\right)$$

- Sometimes, one type of error is worse than the other.
- Asymmetric discrimination = minimise one type of error subject to a bound on the other.

Exact solution

- This problem has been exactly solved!
- Quantum Neyman-Pearson relation:

$$t_p = \|(1-p)\rho_2 - p\rho_1\|_1,$$

$$p\alpha + (1-p)\beta \geq p\alpha^* + (1-p)\beta^* = \frac{1-t_p}{2}$$

- Optimal measurement operator:

$$\Pi_1^* = \{(1-p)\rho_2 - p\rho_1\}_-, \quad \Pi_2^* = \{(1-p)\rho_2 - p\rho_1\}_+$$

So why study this problem further?

- Quantum Neyman-Pearson relation is implicit.
- The trace norm quantity, t_p, can be hard to calculate.
- Difficult to calculate for, in particular: high-dimensional states, multi-copy states, continuous variable states.
- Would like bounds in terms of easily calculable quantities.
- Want to use quantities that are multiplicative across tensor products.
- Want to use quantities that can be easily found for Gaussian states.

Asymptotic limit

- Can study the multi-copy problem in the asymptotic limit.
- Define the asymptotic decay rates.

$$\gamma_\alpha^{\mathcal{T}} = \lim_{N\to\infty} \frac{-\ln\left[\alpha_N^{\mathcal{T}}\right]}{N}, \quad \gamma_\beta^{\mathcal{T}} = \lim_{N\to\infty} \frac{-\ln\left[\beta_N^{\mathcal{T}}\right]}{N}$$

- Quantum Stein's lemma.

$$\sup_{\{\mathcal{T}\}} \left\{\gamma_\alpha^{\mathcal{T}} | \gamma_\beta^{\mathcal{T}} \geq 0\right\} = S_{21}, \qquad \sup_{\{\mathcal{T}\}} \left\{\gamma_\alpha^{\mathcal{T}} | \gamma_\beta^{\mathcal{T}} = S_{12}\right\} = 0$$

- Quantum Hoeffding bound.

$$\sup_{\{\mathcal{T}\}} \left\{\gamma_\alpha^{\mathcal{T}} | \gamma_\beta^{\mathcal{T}} \geq r\right\} = \sup_{0\leq s<1} \frac{-sr - \ln[Q_{s,(1)}]}{1-s}$$

Exact receiver operating characteristic (ROC)

- Can rephrase the quantum Neyman-Pearson relation to get the ROC.

$$\alpha^* = \frac{1-t_p}{2} - \frac{1-p}{2}\frac{dt_p}{dp}, \quad \beta^* = \frac{1-t_p}{2} + \frac{p}{2}\frac{dt_p}{dp}$$

- Assumes differentiable t_p.
- Can replace the gradient with the subgradient.
- We can upper bound and lower bound the ROC using upper and lower bounds on t_p.
- Sometimes, after substituting in an expression for t_p, we can eliminate p.

Fidelity-based bounds

- Trace norm, T, is bounded by the Fuchs-van de Graaf inequalities.
- We can formulate similar bounds on t_p.

$$1 - 2\sqrt{p(1-p)}F(\rho_1, \rho_2) \leq t_p \leq \sqrt{1 - 4p(1-p)F(\rho_1, \rho_2)^2}$$

- The upper bound is exact for pure states.
- This gives bounds on the ROC.

$$\alpha^{(\mathrm{UB},F)} = \frac{1}{4}F^2\beta^{-1}, \quad \alpha^{(\mathrm{LB},F)} = \beta - 2\beta F^2 + F\left(F - 2\sqrt{(1-\beta)\beta(1-F^2)}\right)$$

- The upper bound diverges for β close to either 0 or 1.
- Can modify it to give a piecewise bound.

Quantum Chernoff bounds

- Trace norm is bounded by the QCB.

$$Q_s(\rho_1, \rho_2) = \mathrm{Tr}\left[\rho_2^s \rho_1^{1-s}\right], \quad Q_* = Q_{s_*}, \quad s_* = \mathrm{argmin}_{0 \le s \le 1} Q_s$$

- We can formulate similar bounds on t_p.

$$t_p \geq 1 - 2p^{1-s}(1-p)^s Q_s$$

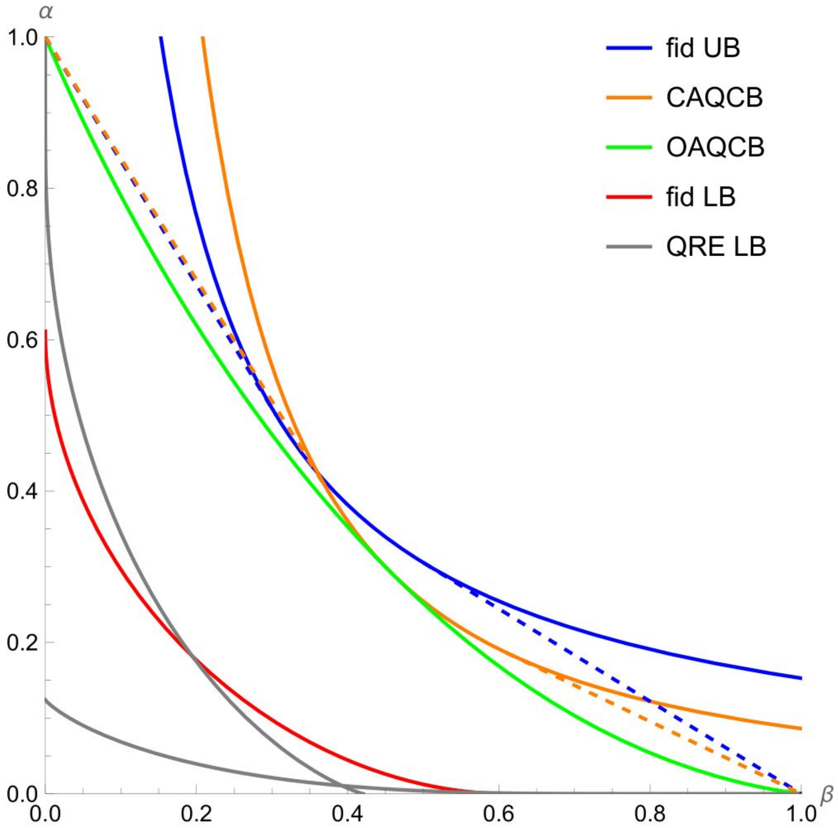
- Constant asymmetric quantum Chernoff bound (CAQCB).

$$\alpha^{(\mathrm{UB}, s_0)} = (1 - s_0) Q_{s_0}^{\frac{1}{1-s_0}} \left(\frac{s_0}{\beta}\right)^{\frac{s_0}{1-s_0}}$$

- Optimal asymmetric quantum Chernoff bound (OAQCB).

$$\alpha^{(\mathrm{UB,QCB})} = \exp\left[-pQ_p^{-1}\frac{dQ_p}{dp}\right](1-p)Q_p, \qquad \beta^{(\mathrm{UB,QCB})} = \exp\left[(1-p)Q_p^{-1}\frac{dQ_p}{dp}\right]pQ_p$$

Comparison of the bounds



ROC for discriminating between a pair of states, each the result of transmitting one mode of a two-mode squeezed vacuum, with an average photon number (per mode) of 4, through a thermal loss channel. ρ_1 (ρ_2) is obtained using a channel with a transmissivity of 0.7 (0.3) and a thermal number of 0.4 (0.6).

Multicopy scaling

- Fidelity and Q_s are both multiplicative across tensor products.

$$F_{(N)} = F_{(1)}^N, \quad Q_{s,(N)} = Q_{s,(1)}^N$$

- The OAQCB for N copies can be expressed in terms of single copy quantities.

$$\alpha_{(N)}^{(\mathrm{UB,QCB})} = \frac{\left(\alpha_{(1)}^{(\mathrm{UB,QCB})}\right)^N}{(1-p)^{N-1}}, \quad \beta_{(N)}^{(\mathrm{UB,QCB})} = \frac{\left(\beta_{(1)}^{(\mathrm{UB,QCB})}\right)^N}{p^{N-1}}$$

- The OAQCB satisfies the quantum Hoeffding bound/quantum Stein's lemma.

$$\gamma_\alpha^{(\mathrm{UB,QCB})} = pQ_{p,(1)}^{-1}\frac{dQ_{p,(1)}}{dp} - \ln\left[Q_{p,(1)}\right], \quad \gamma_\beta^{(\mathrm{UB,QCB})} = -(1-p)Q_{p,(1)}^{-1}\frac{dQ_{p,(1)}}{dp} - \ln\left[Q_{p,(1)}\right]$$

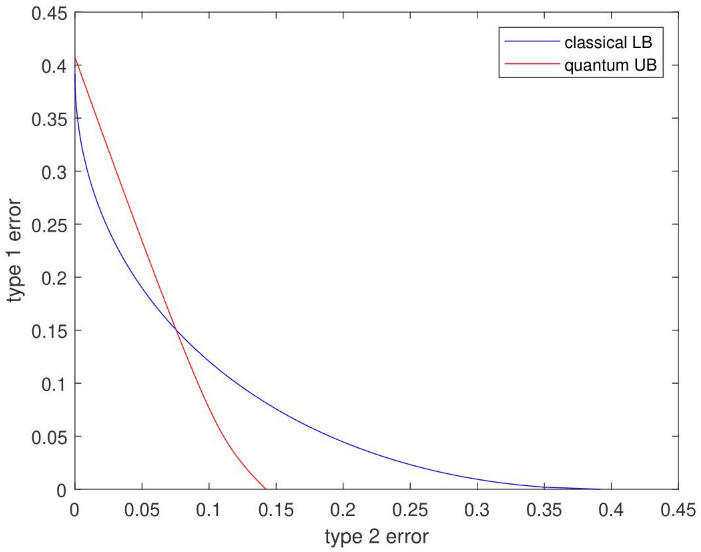
Application examples

- Demonstrating the optimality of adaptive measurement sequences for pure states.

$$\rho_i = \rho_{i,1} \otimes \rho_{i,2}, \quad F_j = F(\rho_{1,j}, \rho_{2,j})$$

- Using the fidelity lower bound, we show this can be achieved with separable measurements, where the second measurement depends on the first result.

$$p_1^{\mp} = \frac{1}{2}\left(1 \mp \sqrt{1 - 4p_0(1-p_0)F_1^2}\right)$$



- Proving quantum advantage.
- Classical/quantum protocols with average photon number 8, discriminating between pure loss channels with transmissivities of 0.95 and 0.4.

Conclusions

- We give an explicit expression for the ROC in terms of t_p.
- We bound t_p in terms of easily calculable quantities (fidelity and Q_s).
- The fidelity lower bound is tight for pure states.
- The OAQCB scales optimally for multiple copies.